Finite Math - Fall 2016

Lecture Notes - 10/12/2016

Section 4.3 - Gauss-Jordan Elimination

Reduced Matrices. In the last section, our goal was to reduce matrices to one of the following forms

$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & p \end{array}\right]$$

where m, n, p are real numbers and $p \neq 0$.

These are all examples of *reduced matrices*, or *reduced row echelon form* matrices. If the matrix has a larger size, we can still put it in reduced form, but it is hard to list out all the possibilities, so we will give a definition here

Definition 1 (Reduced Form)**.** *A matrix is in* reduced form *if*

(1) *Each row consisting entirely of zeros is below any row having at least one nonzero element.*

(2) *The leftmost nonzero element in each row is* 1*.*

(3) *All other elements in the column containing the leftmost* 1 *of a given row are zeros.*

(4) *The leftmost* 1 *in any row is to the right of the leftmost* 1 *in the row above.*

Here are a few examples of matrices in reduced form

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 4 & 8 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Example 1. *Why are the following matrices not in reduced form? Put them in reduced form:*

(a)

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 3 & -6 \end{array}\right]$$

(b)

$$\left[\begin{array}{ccc|c} 1 & 5 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(c)

$$\left[\begin{array}{ccc:c} 0 & 1 & 0 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

(d)

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Solution.

(a) *Here, we do not have a* 1 *in the bottom left, we have a* 3*. Just divide the second row by* 3*,* $\frac{1}{3}R_2 \to R_2$

$$\left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & -2 \end{array}\right]$$

(b) *In this one, there is not zeros above the* 1 *in the second row, second column. To fix this, we use* $R_1 - 5R_2 \to R_1$

$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(c) *Here, the* 1*'s are in the wrong place. To fix it, use* $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

(d) *Here, the row with all zeros isn't below all the other rows, so to fix it, we use* $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let's now actually use Gauss-Jordan elimination to solve a system

Example 2. *Solve the following system using Gauss-Jordan elimination:*

$$\begin{array}{ccccccc} 3x & + & y & - & 2z & = & 2 \\ x & - & 2y & + & z & = & 3 \\ 2x & - & y & - & 3z & = & 3 \end{array}$$

Solution. *First we turn it into an augmented matrix*

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 2 \\ 1 & -2 & 1 & 3 \\ 2 & -1 & -3 & 3 \end{array}\right]$$

Now we follow the process to get a reduced form. Start by getting a 1 *in the top left:*

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 2 \\ 1 & -2 & 1 & 3 \\ 2 & -1 & -3 & 3 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 3 & 1 & -2 & 2 \\ 2 & -1 & -3 & 3 \end{array}\right]$$

and now get zeros everywhere else in that column

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 3 & 1 & -2 & 2 \\ 2 & -1 & -3 & 3 \end{array}\right] \overset{R_2 - 3R_1 \to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 7 & -5 & -7 \\ 2 & -1 & -3 & 3 \end{array}\right] \overset{R_3 - 2R_1 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 7 & -5 & -7 \\ 0 & 3 & -5 & -3 \end{array}\right]$$

Now we need to get a 1 *in the second column, second row*

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 7 & -5 & -7 \\ 0 & 3 & -5 & -3 \end{array}\right] \overset{R_2 - 2R_3 \to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 5 & -1 \\ 0 & 3 & -5 & -3 \end{array}\right]$$

And now we will get 0*'s in the other entries in the second column*

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 5 & -1 \\ 0 & 3 & -5 & -3 \end{array}\right] \overset{R_1 + 2R_2 \to R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 11 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 3 & -5 & -3 \end{array}\right] \overset{R_3 - 3R_2 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 11 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -20 & 0 \end{array}\right]$$

Now, we get a 1 *in the bottom left*

$$\left[\begin{array}{ccc|c} 1 & 0 & 11 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & -20 & 0 \end{array}\right] \overset{-\frac{1}{20}R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 11 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

And finally we get 0*'s everywhere else in that column*

$$\left[\begin{array}{ccc|c} 1 & 0 & 11 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{R_1 - 11R_3 \to R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{R_2 - 5R_3 \to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So, the solution is $x = 1$, $y = -1$, *and* $z = 0$.

Example 3. *Solve by Gauss-Jordan elimination:*

$$\begin{array}{rcrcrcr} 2x_1 & - & 4x_2 & - & x_3 & = & -8 \\ 4x_1 & - & 8x_2 & + & 3x_3 & = & 4 \\ -2x_1 & + & 4x_2 & + & x_3 & = & 11 \end{array}$$

Solution. *Begin by finding the augmented matrix*

$$\left[\begin{array}{ccc|c} 2 & -4 & -1 & -8 \\ 4 & -8 & 3 & 4 \\ -2 & 4 & 1 & 11 \end{array}\right]$$

Now we get the 1 in the upper left

$$\left[\begin{array}{ccc|c} 2 & -4 & -1 & -8 \\ 4 & -8 & 3 & 4 \\ -2 & 4 & 1 & 11 \end{array}\right] \overset{\frac{1}{2}R_1 \to R_1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & -4 \\ 4 & -8 & 3 & 4 \\ -2 & 4 & 1 & 11 \end{array}\right]$$

Now we get the zeros in the rest of the column

$$\left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & -4 \\ 4 & -8 & 3 & 4 \\ -2 & 4 & 1 & 11 \end{array}\right] \overset{R_2-4R_1 \to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & -4 \\ 0 & 0 & 5 & 20 \\ -2 & 4 & 1 & 11 \end{array}\right] \overset{R_3+2R_1 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

The last row of the matrix now corresponds to an equation of the form $0 = 3$, which is nonsense. Thus this system has no solution.

Example 4. *Solve by Gauss-Jordan elimination:*

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & x_3 & = & -7 \\ x_1 & + & x_2 & + & x_3 & = & -1 \\ 2x_1 & & & + & 11x_3 & = & 7 \end{array}$$

Solution. $x_1 = -2, x_2 = 0, x_3 = 1$

Example 5. *Solve by Gauss-Jordan elimination:*

$$\begin{array}{rcrcrcr} 3x_1 & - & 4x_2 & - & x_3 & = & 1 \\ 2x_1 & - & 3x_2 & + & x_3 & = & 1 \\ x_1 & - & 2x_2 & + & 3x_3 & = & 2 \end{array}$$

Solution. *No solution.*

Example 6. *Solve by Gauss-Jordan elimination:*

$$\begin{array}{rcrcrcr} 3x_1 & - & 4x_2 & - & x_3 & = & 1 \\ 2x_1 & - & 3x_2 & + & x_3 & = & 1 \\ x_1 & - & 2x_2 & + & 3x_3 & = & 2 \end{array}$$

Solution. $x_1 = t - 1, x_2 = 2t + 2, x_3 = t$

Example 7. *Solve by Gauss-Jordan elimination:*

$$\begin{array}{rcrcrcr} 2x & - & y & - & 3z & = & 8 \\ x & - & 2y & & & = & 7 \end{array}$$

Solution. *The augmented matrix is*

$$\left[\begin{array}{ccc:c} 2 & -1 & -3 & 8 \\ 1 & -2 & 0 & 7 \end{array}\right]$$

Begin as always, by getting the 1 *in the top left*

$$\left[\begin{array}{ccc:c} 2 & -1 & -3 & 8 \\ 1 & -2 & 0 & 7 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{array}\right]$$

Then getting the zero below it

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{array}\right] \overset{R_2 - 2R_1 \to R_2}{\sim} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 0 & 3 & -3 & -6 \end{array}\right]$$

Now we get the 1 *in the second column*

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 0 & 3 & -3 & -6 \end{array}\right] \overset{\frac{1}{3}R_2 \to R_2}{\sim} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

then use this to get a zero above it

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 0 & 1 & -1 & -2 \end{array}\right] \overset{R1 + 2R_2 \to R_1}{\sim} \left[\begin{array}{ccc:c} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

This tells us that $x - 2z = 3$ *and* $y - z = -2$. *Since* z *is in both equations, we will let* $z = t$, *then we have* $x = 2t + 3$ *and* $y = t - 2$. *So the solutions is*

$$x = 2t + 3, y = t - 2, z = t$$

for real numbers t.

Example 8. *Solve by Gauss-Jordan elimination:*

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 6x_3 & = & 10 \\ 3x_1 & + & 3x_2 & - & 3x_3 & = & 6 \end{array}$$

Solution. $x_1 = -t - 1, x_2 = 2t + 3, x_3 = t$

Example 9. *A company that rents small moving trucks wants to purchase* 16 *trucks with a combined capacity of* 19,200 *cubic feet. Three different types of trucks are available: a cargo van with a capacity of* 300 *cubic feet, a* 15*-foot truck with a capacity of* 900 *cubic feet, and a* 24*-foot truck with a capacity of* 1,500*-cubic feet. How man of each type should the company purchase?*

Solution. $t - 8$ *cargo vans,* $-2t + 24$ *of the* 15*-foot trucks, and* t *of the* 24 *foot trucks, where* $t = 8, 9, 10, 11,$ *or* 12